

Decomposition of Torsion Pairs on Module Categories

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Abstract: In this article, we generalize the concept of torsion pairs and study its structure. As a trial of obtaining all torsion pairs, we decompose torsion pairs by projective modules and injective modules. Then we calculate torsion pairs on the algebra KA_n and tub categories. At last we try to find all torsion pairs on the module categories of finite dimensional hereditary algebras.

Key words: n -torsion pair, n -torsion pair series, 1-type part partition, 2-type part partition, Ext-projective, Ext-injective.

1 Introduction

The concept of torsion pair on abelian category was introduced by Dickson in 1966 [D]. From that time on, torsion pair has been always a useful tool for studying the structure of module categories. However, it seems there is no useful way to find all torsion pairs of a given algebra, although indeed there are some ways to construct torsion pairs among which the most well known is the tilting theory. As a trial, we try to give a way to obtain all torsion pairs of hereditary algebras in this article. This topic is also discussed by Assem and Kerner in [AK] where their most interest is to classify and characterize the torsion pairs by partial tilting modules.

In section 2, we study the general theory where we introduce n -torsion pair and n -torsion pair series as the generalization of classic torsion pair and study its structure. We can see that these two generalizations are essentially the same. In the rest of the paper we would know it is necessary and natural to put forward this conception for studying the structure of torsion pairs. The main skill in this section is from [R] and [TB] where they study HN-filtration for some categories.

There are really a lot of examples to illustrate the necessity to study this finer structure of module categories. For example perpendicular category is obtained by a 2-torsion pair series, and the structure of partial tilting modules can be considered in this way. And HN-filtration can be seen as a generalized n – torsion pair.

In [AK], Assem and Kerner show a relation between some particular partial tilting modules and torsion pairs. In section 3, we adopt their ways by restricting to projective modules and injective modules to try to decompose all torsion pairs. And this is also an application of theories developed in section 2. We give a method for how to decompose a classic torsion pair to n -torsion pairs, and we give a one to one correspondence between all the torsion pairs and some special n -torsion pair on the module category of any artin algebra.

In section 4, we apply the theory in section 3 to path algebras. As a application, we give all the torsion pairs on path algebra KA_n and tube categories. Some of the results also have been shown in [BBM] and [BK]. But we think our results will be much more clear in some aspects.

The section 5 is devoted to obtain all torsion pairs of hereditary algebras which is our purpose. We define an operation called the translation of torsion pairs. Combining this with the operation developed in section 3 and 4, the issue of obtaining all torsion pairs comes down to find all torsion pairs on regular component. For tame hereditary algebras, this problem is equivalent to calculate all torsion pairs on the tube categories in section 4.

We should admit that our way of obtaining all torsion pairs is not very satisfactory since it is mixed with DTr-translation and the extension between different parts of n -torsion pairs.

If there is no special instruction, all modules are left finitely generated modules. For an artin algebra Λ , we denote by $\Lambda\text{-mod}$ the category of all left finitely generated Λ -modules. Subcategories are always assumed to be closed under isomorphism.

2 n – torsion pair and n – torsion pair series

In this section, we assume that Λ is an artin algebra and \mathcal{C} is an extension-closed full subcategory of $\Lambda\text{-mod}$. If $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ are full subcategories of $\Lambda\text{-mod}$, then we denote the minimal full extension-closed subcategory containing $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ by $\langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$. If \mathcal{D} is a subcategory of $\Lambda\text{-mod}$, then we denote the set $\{M \mid \text{Hom}(M, N) = 0, \forall N \in \mathcal{D}\}$ by ${}^{\perp}\mathcal{D}$, the set $\{N \mid \text{Hom}(M, N) = 0, \forall M \in \mathcal{D}\}$ by \mathcal{D}^{\perp} .

The following definition is well known but different from that in [ASS].

Definition 2.1. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of \mathcal{C} is called a torsion pair on \mathcal{C} if the following conditions are satisfied:

- (1) $\text{Hom}(X, Y) = 0$ for all $X \in \mathcal{T}, Y \in \mathcal{F}$.
- (2) $\forall X \in \mathcal{C}$, there exists an exact sequence on $\Lambda\text{-mod}$:

$$0 \longrightarrow X_{\mathcal{T}} \longrightarrow X \longrightarrow X_{\mathcal{F}} \longrightarrow 0$$

such that $X_{\mathcal{T}} \in \mathcal{T}$ and $X_{\mathcal{F}} \in \mathcal{F}$.

Remark 2.2. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on \mathcal{C} . Then $\mathcal{T} = {}^{\perp}\mathcal{F} \cap \mathcal{C}$; $\mathcal{F} = \mathcal{T}^{\perp} \cap \mathcal{C}$; \mathcal{T} and \mathcal{F} are closed under extensions.

Now we give the following definition which is a generalization of the above.

Definition 2.3. an n -tuple $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ of full extension-closed subcategories of \mathcal{C} is called an n -torsion pair if the following conditions are satisfied.

- (1) $\mathcal{C}_i = \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$ for $i = 1, 2, \dots, n+1$.
 - (2) $(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle)$ is a torsion pair on \mathcal{C} for $i = 1, 2, \dots, n+1$.
- Moreover, if the first condition does not satisfy, we call $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ a defect n -torsion pair on \mathcal{C} .

The following lemma is obvious.

Lemma 2.4. Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be 3 full subcategories of $\Lambda\text{-mod}$. Then

- (1) $\langle \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle = \langle \langle \mathcal{C}_1, \mathcal{C}_2 \rangle, \mathcal{C}_3 \rangle = \langle \mathcal{C}_1, \langle \mathcal{C}_2, \mathcal{C}_3 \rangle \rangle$.
- (2) ${}^{\perp}\langle \mathcal{C}_1, \mathcal{C}_2 \rangle = {}^{\perp}\mathcal{C}_1 \cap {}^{\perp}\mathcal{C}_2$, $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle^{\perp} = \mathcal{C}_1^{\perp} \cap \mathcal{C}_2^{\perp}$.
- (3) ${}^{\perp}\langle \mathcal{C}_1 \rangle = {}^{\perp}\mathcal{C}_1, \langle \mathcal{C}_1 \rangle^{\perp} = \mathcal{C}_1^{\perp}$.

Proposition 2.5. Let $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ be an n -torsion pair on \mathcal{C} . If $(\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2, \dots, \tilde{\mathcal{C}}_{k+1})$ be a k -torsion pair on \mathcal{C}_i for some i . Then $(\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{k+1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1})$ is a $(n+k)$ -torsion pair on \mathcal{C} .

Proof: Step 1. If $\mathcal{C}_s \in \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{i-1}\}$, then

$$\begin{aligned} & \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{s-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{s+1}, \dots, \mathcal{C}_{i-1}, \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{k+1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle \\ &= \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{s-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{s+1}, \dots, \mathcal{C}_{i-1}, \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle \\ &= \mathcal{C}_s. \end{aligned}$$

similarly, if $\mathcal{C}_s \in \{\mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1}\}$, then

$$\begin{aligned}
& \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{k+1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{s-1} \rangle^\perp \cap^\perp \langle \mathcal{C}_{s+1}, \dots, \mathcal{C}_{n+1} \rangle \\
&= \mathcal{C}_s. \\
& \text{If } \tilde{\mathcal{C}}_s \in \{\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2, \dots, \tilde{\mathcal{C}}_{k+1}\}, \text{ then} \\
& \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{s-1} \rangle^\perp \cap^\perp \langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1}, \dots, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle \\
&= \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^\perp \cap \langle \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{s-1} \rangle^\perp \cap^\perp \langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1} \rangle \cap^\perp \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle \\
&= \mathcal{C}_i \cap \langle \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{s-1} \rangle^\perp \cap^\perp \langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1} \rangle \\
&= \tilde{\mathcal{C}}_s.
\end{aligned}$$

Thus, the checking of the first condition of definition 2.3 is finished.

Step 2. Without losing of generality, we may assume $1 \leq s \leq k$, and we want to check $(\langle \mathcal{C}_1, \dots, \tilde{\mathcal{C}}_s \rangle, \langle \tilde{\mathcal{C}}_{s+1}, \dots, \mathcal{C}_{n+1} \rangle)$ is a torsion pair on \mathcal{C} .

Given $X \in \mathcal{C}$, because $(\langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle, \langle \mathcal{C}_i, \dots, \mathcal{C}_{n+1} \rangle)$ is a torsion pair on \mathcal{C} , there is an exact sequence

$$0 \longrightarrow X_1 \xrightarrow{i_1} X \xrightarrow{\pi_1} X_2 \longrightarrow 0$$

such that $X_1 \in \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle$ and $X_2 \in \langle \mathcal{C}_i, \dots, \mathcal{C}_{n+1} \rangle$.

By torsion pair $(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle)$, there is an exact sequence

$$0 \longrightarrow X_3 \xrightarrow{i_2} X_2 \xrightarrow{\pi_2} X_4 \longrightarrow 0$$

such that $X_3 \in \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle$ and $X_4 \in \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$.

Because $X_3 \in \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^\perp$ since $X_2 \in \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^\perp$, so $X_3 \in \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^\perp \cap^\perp \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle = \mathcal{C}_i$.

By torsion pair $(\langle \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_s \rangle, \langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1} \rangle)$ on \mathcal{C}_i , there is an exact sequence

$$0 \longrightarrow X_5 \xrightarrow{i_3} X_3 \xrightarrow{\pi_3} X_6 \longrightarrow 0$$

such that $X_5 \in \langle \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_s \rangle$ and $X_6 \in \langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1} \rangle$.

By pushout of i_2 and π_3 , we have the following commutative diagram

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
& X_5 & \longrightarrow & X'_5 & & & \\
& \downarrow i_3 & & \downarrow i_4 & & & \\
0 & \longrightarrow & X_3 & \xrightarrow{i_2} & X_2 & \xrightarrow{\pi_2} & X_4 \longrightarrow 0 \\
& \downarrow \pi_3 & & \downarrow \pi_4 & & \parallel & \\
0 & \longrightarrow & X_6 & \longrightarrow & X_{\mathcal{F}} & \longrightarrow & X_4 \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

By snake lemma, $X_5 = X'_5$, so we have an exact sequence

$$0 \longrightarrow X_5 \xrightarrow{i_4} X_2 \xrightarrow{\pi_4} X_{\mathcal{F}} \longrightarrow 0$$

such that $X_{\mathcal{F}} \in \langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$.

By pullback of i_4 and π_1 , we have the following commutative diagram:

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & X_1 & \longrightarrow & X_{\mathcal{T}} & \longrightarrow & X_5 \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow i_4 \\
0 & \longrightarrow & X_1 & \xrightarrow{i_1} & X & \xrightarrow{\pi_1} & X_2 \longrightarrow 0 \\
& & & \downarrow & & \downarrow \pi_4 & \\
& & & X'_{\mathcal{F}} & \longrightarrow & X_{\mathcal{F}} & \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 &
\end{array}$$

By snake lemma, $X'_{\mathcal{F}} = X_{\mathcal{F}}$, so we have the following exact sequence:

$$0 \longrightarrow X_{\mathcal{T}} \longrightarrow X \longrightarrow X_{\mathcal{F}} \longrightarrow 0$$

such that $X_{\mathcal{T}} \in \langle \mathcal{C}_1, \dots, \tilde{\mathcal{C}}_s \rangle$ and $X_{\mathcal{F}} \in \langle \tilde{\mathcal{C}}_{s+1}, \dots, \mathcal{C}_{n+1} \rangle$.

Now we give the following definition which is very important to learn the structure of n -torsion pair.

Definition 2.6. Series $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ of torsion pairs on \mathcal{C} is called an n -torsion pair series if $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \dots \subseteq \mathcal{T}_n$ (equivalently, $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \dots \supseteq \mathcal{F}_n$).

The following definition is an operation.

Definition 2.7. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on \mathcal{C} , and \mathcal{D} be a subcategory of \mathcal{C} . We call $(D_{(\mathcal{T}, \mathcal{F})}^1(\mathcal{D}), D_{(\mathcal{T}, \mathcal{F})}^2(\mathcal{D}))$ is a decomposition of \mathcal{D} along $(\mathcal{T}, \mathcal{F})$, where $D_{(\mathcal{T}, \mathcal{F})}^1(\mathcal{D}) = \{X \mid \text{There exists an exact sequence } 0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0 \text{ such that } X \in \mathcal{T}, Y \in \mathcal{F}, M \in \mathcal{D}\}$, $D_{(\mathcal{T}, \mathcal{F})}^2(\mathcal{D}) = \{Y \mid \text{There exists an exact sequence } 0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0 \text{ such that } X \in \mathcal{T}, Y \in \mathcal{F}, M \in \mathcal{D}\}$.

Lemma 2.8. If $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2)\}$ is a 2-torsion pair series on \mathcal{C} . Then

$$\mathcal{F}_1 \cap \mathcal{T}_2 = D_{(\mathcal{T}_1, \mathcal{F}_1)}^2(\mathcal{T}_2) = D_{(\mathcal{T}_2, \mathcal{F}_2)}^1(\mathcal{F}_1).$$

Proof: $\mathcal{F}_1 \cap \mathcal{T}_2 \subseteq D_{(\mathcal{T}_1, \mathcal{F}_1)}^2(\mathcal{T}_2)$ is clear.

Suppose $X \in \mathcal{T}_2$, by torsion pair $(\mathcal{T}_1, \mathcal{F}_1)$, there is an exact sequence

$$0 \longrightarrow X_{\mathcal{T}_1} \longrightarrow X \longrightarrow X_{\mathcal{F}_1} \longrightarrow 0$$

such that $X_{\mathcal{T}_1} \in \mathcal{T}_1$ and $X_{\mathcal{F}_1} \in \mathcal{F}_1$.

However, $X_{\mathcal{F}_1} \in {}^\perp \mathcal{F}_2$ since $X \in {}^\perp \mathcal{F}_2$. Thus, $X_{\mathcal{F}_1} \in {}^\perp \mathcal{F}_2 \cap \mathcal{C} = \mathcal{T}_2$ and $X_{\mathcal{F}_1} \in \mathcal{T}_2 \cap \mathcal{F}_1$. So $\mathcal{F}_1 \cap \mathcal{T}_2 = D_{(\mathcal{T}_1, \mathcal{F}_1)}^2(\mathcal{T}_2)$.

The other half is similar.

n -torsion pair series will give a filtration for every module which is demonstrated below.

Proposition 2.9. If $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ is an n -torsion pair series on \mathcal{C} . Then for every module X in \mathcal{C} , there is a filtration:

$$\begin{array}{ccccccc} 0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots \longrightarrow X_{n+1} \xlongequal{\quad} X \\ & & & \searrow & & \searrow & \\ & & & S_1 & & S_{n+1} & \end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence for $i = 1, 2, \dots, n+1$, and $S_1 \in \mathcal{T}_1, S_i \in \mathcal{F}_{i-1} \cap \mathcal{T}_i$ for $1 < i < n+1$, $S_{n+1} \in \mathcal{F}_n$ and $X_j \in \mathcal{T}_j$ for $j < n+1$.

Proof. Using induction on n .

$n = 1$, by the second condition of definition 2.1, there is a filtration

$$\begin{array}{ccccccc} 0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 \xlongequal{\quad} X \\ & & & & \searrow & & \searrow \\ & & & & S_1 & & S_2 \end{array}$$

such that $0 \rightarrow X_1 \rightarrow X_2 \rightarrow S_2 \rightarrow 0$ is an exact sequence and $X_1 \in \mathcal{T}_1, S_2 \in \mathcal{F}_1$.

Suppose that the proposition is true for $n = k$, let us consider $n = k + 1$. By torsion pair $(\mathcal{T}_{k+1}, \mathcal{F}_{k+1})$ on \mathcal{C} , there is an exact sequence

$$0 \longrightarrow X_{k+1} \longrightarrow X \longrightarrow S_{k+2} \longrightarrow 0$$

such that $X_{k+1} \in \mathcal{T}_{k+1}$ and $S_{k+2} \in \mathcal{F}_{k+1}$.

Because $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_k, \mathcal{F}_k)\}$ is a k -torsion pair series on \mathcal{C} , by induction, there is a filtration:

$$\begin{array}{ccccccccccc} 0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_k & \longrightarrow & X_{k+1} \xlongequal{\quad} X_{k+1} \\ & & & & \searrow & & & & \searrow & & \searrow \\ & & & & S_1 & & & & S_k & & S_{k+1} \end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence for $i = 1, 2, \dots, k + 1$, and $S_1 \in \mathcal{T}_1, S_i \in \mathcal{F}_{i-1} \cap \mathcal{T}_i$ for $1 < i < k + 1$, $S_{k+1} \in \mathcal{F}_k, X_i \in \mathcal{T}_i$ for all i . However $S_{k+1} \in \mathcal{T}_{k+1}$ since $X_{k+1} \in \mathcal{T}_{k+1}$. So $S_{k+1} \in \mathcal{F}_k \cap \mathcal{T}_{k+1}$. The filtration is given.

Proposition 2.10. *If $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ is an n -torsion pair series on \mathcal{C} . Then $\mathcal{F}_i \cap \mathcal{T}_{i+k} = \langle \mathcal{F}_i \cap \mathcal{T}_{i+1}, \mathcal{F}_{i+1} \cap \mathcal{T}_{i+2}, \dots, \mathcal{F}_{i+k-1} \cap \mathcal{T}_{i+k} \rangle$.*

Proof. " \supseteq " is obviously.

" \subseteq " : For $X \in \mathcal{F}_i \cap \mathcal{T}_{i+k}$, by the above lemma, there is a filtration of X :

$$\begin{array}{ccccccccccc} 0 & \xlongequal{\quad} & X_0 & \cdots \longrightarrow & X_{i+1} & \longrightarrow & \cdots & \longrightarrow & X_{i+k} & \longrightarrow & \cdots & \longrightarrow & X_{n+1} \xlongequal{\quad} X \\ & & & & \searrow & & & & \searrow & & & & \searrow \\ & & & & S_{i+1} & & & & S_{i+k} & & & & S_{n+1} \end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence for $i = 1, 2, \dots, n + 1$, and $S_1 \in \mathcal{T}_1, S_i \in \mathcal{F}_{i-1} \cap \mathcal{T}_i$ for $1 < i < n + 1$, $S_{n+1} \in \mathcal{F}_n, X_i \in \mathcal{T}_i$ for $i < n + 1$.

First, we claim that $X_0 = X_1 = \cdots = X_i = 0$.

In fact, $\text{Hom}(X_i, X_{i+1}) = 0$ since $X_i \in \mathcal{T}_i$ and X_{i+1} is submodule of X belongs to \mathcal{F}_i . By the exact sequence $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$, one gains $X_i = 0$. Hence $X_0 = X_1 = \cdots = X_{i-1} = 0$.

Second, we claim that $X_{i+k+1} = X_{i+k+2} = \cdots = X_{n+1} = X$.

In fact, $\text{Hom}(X_{n+1}, S_{n+1}) = 0$ since $X_{n+1} = X \in \mathcal{F}_i \cap \mathcal{T}_{i+k}$ and $S_{n+1} \in \mathcal{F}_n$. By exact sequence $0 \rightarrow X_n \rightarrow X_{n+1} \rightarrow S_{n+1} \rightarrow 0$, one gains $S_{n+1} = 0$ and $X_n = X_{n+1} = X$. Similarly, we have $X_{i+k+1} = X_{i+k+2} = \cdots = X_{n-1} = X$.

Now, we have the following filtration:

$$\begin{array}{ccccccc} 0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_{i+1} & \longrightarrow & \cdots \longrightarrow X_{i+k} \longrightarrow X_{i+k+1} \xlongequal{\quad} X \\ & & & & \searrow & & \searrow & & \searrow \\ & & & & S_{i+1} & & S_{i+k} & & S_{i+k+1} \end{array}$$

Thus $X \in \langle \mathcal{F}_i \cap \mathcal{T}_{i+1}, \mathcal{F}_{i+1} \cap \mathcal{T}_{i+2}, \cdots, \mathcal{F}_{i+k-1} \cap \mathcal{T}_{i+k} \rangle$.

The following is the relation between n -torsion pair and n -torsion pair series.

Theorem 2.11. *There is a one to one correspondence between the set of n -torsion pair series on \mathcal{C} and the set of n -torsion pair on \mathcal{C} :*

$$\left\{ \begin{array}{l} (\mathcal{T}_1, \mathcal{F}_1), \cdots, (\mathcal{T}_n, \mathcal{F}_n) \\ n\text{-torsion pair series on } \mathcal{C} \end{array} \right\} \xrightleftharpoons[\beta]{\alpha} \left\{ \begin{array}{l} (\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{n+1}) \\ n\text{-torsion pair on } \mathcal{C} \end{array} \right\}$$

such that $\alpha(\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \cdots, (\mathcal{T}_n, \mathcal{F}_n)\}) = (\mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \cdots, \mathcal{F}_{n-1} \cap \mathcal{T}_n, \mathcal{F}_n)$ and $\beta((\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{n+1})) = \{(\langle \mathcal{C}_1, \cdots, \mathcal{C}_i \rangle, \langle \mathcal{C}_i, \cdots, \mathcal{C}_{n+1} \rangle) \mid i = 1, 2, \cdots, n\}$.

Proof: First, we check that $(\mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \cdots, \mathcal{F}_{n-1} \cap \mathcal{T}_n, \mathcal{F}_n)$ is an n -torsion pair on \mathcal{C} .

(1) $\mathcal{F}_{i-1} \cap \mathcal{T}_i = \mathcal{C} \cap \mathcal{T}_{i-1}^\perp \cap \mathcal{C} \cap {}^\perp \mathcal{F}_i = \mathcal{C} \cap \mathcal{T}_{i-1}^\perp \cap {}^\perp \mathcal{F}_i = \mathcal{C} \cap \langle \mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \cdots, \mathcal{F}_{i-2} \cap \mathcal{T}_{i-1} \rangle^\perp \cap {}^\perp \langle \mathcal{F}_i \cap \mathcal{T}_{i+1}, \cdots, \mathcal{F}_n \rangle$ by the above proposition.

(2) Obviously, $(\langle \mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \cdots, \mathcal{F}_{i-1} \cap \mathcal{T}_i \rangle, \langle \mathcal{F}_i \cap \mathcal{T}_{i+1}, \cdots, \mathcal{F}_n \rangle) = (\mathcal{T}_i, \mathcal{F}_i)$.

Second, we check that $\{(\langle \mathcal{C}_1, \cdots, \mathcal{C}_i \rangle, \langle \mathcal{C}_i, \cdots, \mathcal{C}_{n+1} \rangle)\}_{i=1,2,\dots,n}$ is an n -torsion pair series on \mathcal{C} . But this is clear.

Third, we check that $\beta\alpha = 1$.

$$\begin{aligned} \beta\alpha(\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \cdots, (\mathcal{T}_n, \mathcal{F}_n)\}) &= \beta(\mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \cdots, \mathcal{F}_{n-1} \cap \mathcal{T}_n, \mathcal{F}_n) \\ &= \{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \cdots, (\mathcal{T}_n, \mathcal{F}_n)\} \text{ by the above proposition.} \end{aligned}$$

Last, we check that $\alpha\beta = 1$.

$$\begin{aligned}
\alpha\beta((\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})) &= \alpha(\{(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle) \mid i = 1, 2, \dots, n\}) \\
&= \{\langle \mathcal{C}_i, \dots, \mathcal{C}_{n+1} \rangle \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle \mid i = 1, 2, \dots, n+1\} \\
&= \{\mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^\perp \cap {}^\perp \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle \mid i = 1, 2, \dots, n+1\} \\
&= (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1}).
\end{aligned}$$

Proposition 2.12. $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ is an n -torsion pair on \mathcal{C} if and only if

- (1) $\text{Hom}(X, Y) = 0$ for all $X \in \mathcal{C}_i, Y \in \mathcal{C}_j, i < j$.
- (2) For every $X \in \mathcal{C}$, there is a filtration:

$$\begin{array}{ccccccc}
0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots \longrightarrow X_{n+1} \xlongequal{\quad} X \\
& & & & \searrow & & \searrow \\
& & & & S_1 & & S_{n+1}
\end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence and $S_i \in \mathcal{C}_i$ for all i .

Proof: " \implies ": Let $\mathcal{T}_i = \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \mathcal{F}_i = \langle \mathcal{C}_i, \dots, \mathcal{C}_{n+1} \rangle, i = 1, 2, \dots, n$. Then $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ is an n -torsion pair series by proposition 2.12.

By the proof of the above proposition, we know $\mathcal{C}_i = \mathcal{F}_{i-1} \cap \mathcal{T}_i$.

Hence, for every module X in \mathcal{C} , there is a filtration:

$$\begin{array}{ccccccc}
0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots \longrightarrow X_{n+1} \xlongequal{\quad} X \\
& & & & \searrow & & \searrow \\
& & & & S_1 & & S_{n+1}
\end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence and $S_i \in \mathcal{F}_{i-1} \cap \mathcal{T}_i = \mathcal{C}_i$.

" \impliedby ": First, we show that $\mathcal{C}_i = \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^\perp \cap {}^\perp \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$ for $i = 1, 2, \dots, n+1$.

" \subseteq " is clear;

" \supseteq ": $\forall X \in \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^\perp \cap {}^\perp \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$, there is a filtration:

$$\begin{array}{ccccccccccc}
0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{i-1} & \longrightarrow & X_i & \longrightarrow & X_{i+1} & \longrightarrow & \cdots & \longrightarrow & X_{n+1} \xlongequal{\quad} X \\
& & \searrow & & & & & & \searrow & & \searrow & & \searrow & & & & \searrow \\
& & S_1 & & & & & & S_{i-1} & & S_i & & S_{i+1} & & & & S_{n+1}
\end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence and $S_i \in \mathcal{C}_i$.

Just like the proof of proposition 2.10, we have $X_0 = X_1 = \dots = X_{i-1} = 0$ and $X_i = X_{i+1} = \dots = X_{n+1} = X$. So $X_i = S_i \in \mathcal{C}_i$.

Second, we show that $(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle)$ is a torsion pair on \mathcal{C} by definition 2.1:

- (1) Clear!
- (2) $\forall X \in \mathcal{C}$, there is a filtration:

$$\begin{array}{ccccccccccc}
 0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{i-1} & \longrightarrow & X_i & \longrightarrow & X_{i+1} & \longrightarrow & \cdots & \longrightarrow & X_{n+1} & \xlongequal{\quad} & X \\
 & & & & \searrow & & & & \searrow & & \searrow & & \searrow & & & & \searrow & & \\
 & & & & S_1 & & & & S_{i-1} & & S_i & & S_{i+1} & & & & S_{n+1} & &
 \end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence and $S_i \in \mathcal{C}_i$.

It is clear that $X_i \in \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle$, we claim that $X/X_i \in \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$.

In fact, by snake lemma we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X_i & \longrightarrow & X_{i+1} & \longrightarrow & S_{i+1} \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X_i & \longrightarrow & X_{i+2} & \longrightarrow & X_{i+2}/X_i \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & S_{i+2} & \xlongequal{\quad} & S_{i+2} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Hence $X_{i+2}/X_i \in \langle \mathcal{C}_{i+1}, \mathcal{C}_{i+2} \rangle$.

Use snake lemma again, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X_i & \longrightarrow & X_{i+2} & \longrightarrow & X_{i+2}/X_i \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X_i & \longrightarrow & X_{i+3} & \longrightarrow & X_{i+3}/X_i \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & S_{i+3} & \xlongequal{\quad} & S_{i+3} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Hence $X_{i+3}/X_i \in \langle \mathcal{C}_{i+1}, \mathcal{C}_{i+2}, \mathcal{C}_{i+3} \rangle$.

Similarly, we can obtain $X_{n+1}/X_i \in \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$.

Now, $0 \rightarrow X_i \rightarrow X_{n+1} \rightarrow X_{n+1}/X_i \rightarrow 0$ is the desired exact sequence.

The following lemma is well known[D].

Lemma 2.13. *If \mathcal{B} is a subcategory of $\Lambda\text{-mod}$, then ${}^\perp({}^\perp\mathcal{B})^\perp = {}^\perp\mathcal{B}$ and $({}^\perp(\mathcal{B}^\perp))^\perp = \mathcal{B}^\perp$ and $({}^\perp\mathcal{B}, ({}^\perp\mathcal{B})^\perp)$ and $(\mathcal{B}^\perp, {}^\perp(\mathcal{B}^\perp))$ are both torsion pairs.*

The following means that the condition (2) in Definition 2.3 will be superfluous in some conditions.

Corollary 2.14. *Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ be full subcategories of $\Lambda\text{-mod}$, if $\mathcal{C}_i = \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^\perp \cap {}^\perp\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$ for $i = 1, 2, \dots, n+1$. Then $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)$ is an n -torsion pair on $\Lambda\text{-mod}$.*

Proof: It is enough to show the second condition of the above proposition since the first condition is clear.

By the above lemma, there is a fact: $({}^\perp\mathcal{C}_{n+1}, \mathcal{C}_{n+1})$ is a torsion pair since $\mathcal{C}_{n+1} = \langle \mathcal{C}_1, \dots, \mathcal{C}_n \rangle^\perp$.

Now, we use induction on n to show.

If $n = 1$, clear.

Suppose that the proposition is true for $n = k \geq 1$, we consider the case of $n = k+1$.

Step 1, claim: $\langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle = \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^\perp$.

In fact, " \subseteq " is clear.

" \supseteq ": $\forall X \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^\perp$, by torsion pair $({}^\perp\mathcal{C}_{k+2}, \mathcal{C}_{k+2})$, \exists an exact sequence $0 \rightarrow X_{k+1} \rightarrow X \rightarrow T_{k+2} \rightarrow 0$ such that $X_{k+1} \in {}^\perp\mathcal{C}_{k+2}$ and $T_{k+2} \in \mathcal{C}_{k+2}$. $X_{k+1} \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^\perp$ since $X \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^\perp$. Thus $X_{k+1} \in \mathcal{C}_{k+1}$ and $X \in \langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle$.

Step 2. By induction, $(\mathcal{C}_1, \dots, \mathcal{C}_k, \langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle)$ is a k -torsion pair on $\Lambda\text{-mod}$. So $\forall X \in \Lambda\text{-mod}$, there is a filtration:

$$\begin{array}{ccccccc}
 0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots \longrightarrow X_{k-1} \longrightarrow X_k \longrightarrow X_{k+1} \xlongequal{\quad} X \\
 & & & \searrow & & \searrow & \searrow & \searrow & \searrow \\
 & & & S_1 & & S_{k-1} & & S_k & & S
 \end{array}$$

such that $S_i \in \mathcal{C}_i$ and $S \in \langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle$.

By torsion pair $({}^\perp\mathcal{C}_{k+2}, \mathcal{C}_{k+2})$, there is an exact sequence $0 \rightarrow S_{k+1} \rightarrow S \rightarrow S_{k+2} \rightarrow 0$ such that $S_{k+1} \in {}^\perp\mathcal{C}_{k+2}$ and $S_{k+2} \in \mathcal{C}_{k+2}$.

Because $S \in \langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle$, then $S \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^\perp$, so $S_{k+1} \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^\perp$, hence $S_{k+1} \in \mathcal{C}_{k+1}$ since $S_{k+1} \in {}^\perp\mathcal{C}_{k+2}$.

By pullback of $(X \rightarrow S, S_{k+1} \rightarrow S)$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_k & \longrightarrow & X_{k+1} & \longrightarrow & S_{k+1} \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_k & \longrightarrow & X & \longrightarrow & S \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & S_{k+2} & = & S_{k+2} \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

Now, we find a filtration:

$$\begin{array}{ccccccccccc}
0 & = & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{k-1} & \longrightarrow & X_k & \longrightarrow & X_{k+1} & \longrightarrow & X_{k+2} & = & X \\
& & & & \swarrow & & & & \swarrow & & \swarrow & & \swarrow & & \swarrow & & \\
& & & & S_1 & & & & S_{k-1} & & S_k & & S_{k+1} & & S_{k+2} & &
\end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence and $S_i \in \mathcal{C}_i$.

Proposition 2.15. *Let $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)$ is an n -torsion pair on \mathcal{C} . Then*

- (1) $\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle = \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle^\perp \cap {}^\perp \langle \mathcal{C}_{i+k+1}, \dots, \mathcal{C}_{n+1} \rangle$
- (2) $(\mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k})$ is a k -torsion pair on $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle$
- (3) $(\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \langle \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle, \mathcal{C}_{i+k+1}, \dots, \mathcal{C}_{n+1})$ is an $(n-k)$ -torsion pair.

Proof. (1) " \subseteq ": clear!

" \supseteq ": $\forall X \in \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle^\perp \cap {}^\perp \langle \mathcal{C}_{i+k+1}, \dots, \mathcal{C}_{n+1} \rangle$, there is a filtration:

$$\begin{array}{ccccccc}
0 & = & X_0 & \cdots \longrightarrow & X_{i+1} & \longrightarrow & \cdots \longrightarrow & X_{i+k} & \longrightarrow & \cdots \longrightarrow & X_{n+1} & = & X \\
& & & & \swarrow & & & \swarrow & & & \swarrow & & \\
& & & & S_{i+1} & & & S_{i+k} & & & S_{n+1} & &
\end{array}$$

such that $X_0 = X_1 = \cdots = X_{i-1} = 0$ and $X_{i+k+1} = X_{i+k+2} = \cdots = X_{n+1} = X$.

(2) Checking by Definition 2.3, the first condition holds by (1), and the second condition holds by similar techniques in proof of proposition 2.10 and (1).

(3) Checking by Definition 2.3, the first condition obviously holds, $\forall X \in \mathcal{C}$, there is a filtration:

$$\begin{array}{ccccccc} 0 & \xlongequal{\quad} & X_0 & \cdots & \longrightarrow & X_{i+1} & \longrightarrow \cdots \longrightarrow X_{i+k} & \longrightarrow \cdots \longrightarrow X_{n+1} & \xlongequal{\quad} & X \\ & & & & & \swarrow & \swarrow & \swarrow & & \\ & & & & & S_{i+1} & S_{i+k} & S_{n+1} & & \end{array}$$

use the similar techniques in the last part of proof of proposition 2.12, we have the following exact sequence:

$$0 \longrightarrow X_i \longrightarrow X_{i+k} \longrightarrow X_{i+k}/X_i \longrightarrow 0$$

Let $\hat{S} = X_{i+k}/X_i$, then we have the desired filtration:

$$\begin{array}{ccccccc} 0 & \xlongequal{\quad} & X_0 & \cdots & \longrightarrow & X_{i+1} & \longrightarrow X_{i+k} & \longrightarrow \cdots \longrightarrow X_{n+1} & \xlongequal{\quad} & X \\ & & & & & \swarrow & \swarrow & \swarrow & & \\ & & & & & S_{i+1} & \hat{S} & S_{n+1} & & \end{array}$$

Corollary 2.16. *Suppose $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ is an n -torsion pair series on \mathcal{C} . Let $\mathcal{F}_0 = \mathcal{T}_{n+1} = \mathcal{C}, \mathcal{F}_{n+1} = \mathcal{T}_0 = 0$. Then $\{(\mathcal{T}_{i+1} \cap \mathcal{F}_i, \mathcal{F}_{i+1} \cap \mathcal{T}_{i+k+1}), \dots, (\mathcal{T}_{i+k} \cap \mathcal{F}_i, \mathcal{F}_{i+k} \cap \mathcal{T}_{i+k+1})\}$ is a k -torsion pair series on $\mathcal{T}_{i+k+1} \cap \mathcal{F}_i$ for $i = 0, 1, \dots, n-1$ and $k > 0$.*

Proof. Let $\mathcal{C}_1 = \mathcal{T}_1, \mathcal{C}_l = \mathcal{F}_{l-1} \cap \mathcal{T}_l (2 \leq l \leq n), \mathcal{C}_{n+1} = \mathcal{F}_{n+1}$. So $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ is an n -torsion pair on \mathcal{C} , and $(\mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k})$ is a k -torsion pair on $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle$. Thus $\{(\langle \mathcal{C}_i, \dots, \mathcal{C}_{i+l} \rangle, \langle \mathcal{C}_{i+l+1}, \dots, \mathcal{C}_{i+k+l} \rangle) \mid l = 1, 2, \dots, k\}$ is a k -torsion pair series on $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle$. But $\langle \mathcal{C}_i, \dots, \mathcal{C}_{i+l} \rangle = \mathcal{F}_i \cap \mathcal{T}_{i+l}, \langle \mathcal{C}_{i+l+1}, \dots, \mathcal{C}_{i+k+l} \rangle = \mathcal{F}_{i+1} \cap \mathcal{T}_{i+k+l}$. The corollary is proved.

Corollary 2.17. *If $(\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{n+1})$ is a defect n -torsion pair on \mathcal{C} . Then there is an unique n -torsion pair $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ on \mathcal{C} such that $\mathcal{D}_i \subseteq \mathcal{C}_i$.*

Proof. Let $\mathcal{T}_i = \langle \mathcal{D}_1, \dots, \mathcal{D}_i \rangle, \mathcal{F}_i = \langle \mathcal{D}_{i+1}, \dots, \mathcal{D}_n \rangle$, Then $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ is an n -torsion pair series on \mathcal{C} .

Let $\mathcal{C}_i = \mathcal{F}_{i-1} \cap \mathcal{T}_i$, then $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ is an n -torsion pair on \mathcal{C} such that $\mathcal{D}_i \subseteq \mathcal{C}_i$.

Suppose $(\mathcal{C}'_1, \mathcal{C}'_2, \dots, \mathcal{C}'_{n+1})$ is an other n – torsion pair on \mathcal{C} such that $\mathcal{D}_i \subseteq \mathcal{C}'_i$, then $\mathcal{T}_i = \langle \mathcal{D}_1, \dots, \mathcal{D}_i \rangle \subseteq \langle \mathcal{C}'_1, \dots, \mathcal{C}'_i \rangle = \mathcal{T}'$, Similarly, $\mathcal{F} \subseteq \mathcal{F}'$. Therefore, $\mathcal{C}_i = \mathcal{F}_{i-1} \cap \mathcal{T}_i = \mathcal{F}'_{i-1} \cap \mathcal{T}'_i = \mathcal{C}'_i$.

The following proposition is very useful.

Proposition 2.18. *Suppose $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2)\}$ is a 2 – torsion pair series on \mathcal{C} . Then we have the following 1 to 1 correspondence :*

$$\{(\mathcal{T}', \mathcal{F}'): 1 - \text{torsion pair on } \mathcal{F}_1 \cap \mathcal{T}_2\} \xrightleftharpoons[G]{F} \{(\mathcal{T}_3, \mathcal{F}_3): 1 - \text{torsion pair on } \mathcal{C} \text{ such that } \mathcal{T}_1 \subseteq \mathcal{T}_3 \subseteq \mathcal{T}_2\}$$

where $F((\mathcal{T}', \mathcal{F}')) = (\langle \mathcal{T}_1, \mathcal{T}' \rangle, \langle \mathcal{F}', \mathcal{F}_2 \rangle)$, $G((\mathcal{T}_3, \mathcal{F}_3)) = (\mathcal{T}_3 \cap \mathcal{F}_1, \mathcal{F}_3 \cap \mathcal{T}_2)$.

Proof: By Proposition 2.5, Theorem 2.11 and Proposition 2.15, it is clear.

Remark 2.19. *The above lemma has a lot of generalized forms since we have so many results. And those forms can give a finer characterization for torsion pairs and module categories. For example, Theorem 2.1 in [AK].*

The following is an example of n – torsion pair.

Example 2.20. *Let T be a tilting module, T_1, T_2, \dots, T_n be all non-isomorphic indecomposable summands of T . Then*

$$(\text{Gen}(T_1), T_1^\perp \bigcap \text{Gen}(T_1 \oplus T_2), \dots, (T_1 \oplus \dots \oplus T_{n-1})^\perp \bigcap \text{Gen}(T))$$

is an $(n - 1)$ – torsion pair on $\text{Gen}(T)$.

Proof. Let $X_i = T_1 \oplus \dots \oplus T_i$, then $(\text{Gen} X_i, X_i^\perp)$ is a torsion pair on $\Lambda\text{-mod}$. And $\{(\text{Gen} X_i, X_i^\perp) \mid i = 1, 2, \dots, n\}$ is an n – torsion pair series on $\Lambda\text{-mod}$. Therefore,

$$(\text{Gen}(T_1), T_1^\perp \bigcap \text{Gen}(T_1 \oplus T_2), \dots, (T_1 \oplus \dots \oplus T_{n-1})^\perp \bigcap \text{Gen}(T_n), T^\perp)$$

is an n – torsion pair on $\Lambda\text{-mod}$ by Theorem 2.11. So $(\text{Gen}(T_1), T_1^\perp \bigcap \text{Gen}(T_1 \oplus T_2), \dots, (T_1 \oplus \dots \oplus T_{n-1})^\perp \bigcap \text{Gen}(T))$ is an $(n - 1)$ – torsion pair on $\text{Gen}(T)$ by Proposition 2.15.

3 Decomposition by projective and injective modules

In this section, we always suppose Λ is an artin algebra. For given artin algebra Γ , we denote: $\mathcal{P}(\Gamma)$ is the category of all projective modules in $\Gamma\text{-mod}$, $\mathcal{I}(\Gamma)$ is

the category of all injective modules in $\Gamma\text{-mod}$; $\mathbf{E}(\Gamma) = \{(\mathcal{T}, \mathcal{F})\}$ is torsion pair on $\Gamma\text{-mod} \mid \mathcal{T} \cap \mathcal{P}(\Gamma) = \mathcal{F} \cap \mathcal{I}(\Gamma) = \emptyset\}$. For a set Ψ we denote the number of the elements of Ψ by $\#\Psi$. For a subcategory \mathcal{D} of $\Lambda\text{-mod}$, let $\text{Ind } \mathcal{D}$ be the set of pairwise non-isomorphic indecomposable modules in \mathcal{D} . For a module M , let $\text{Ind } M = \text{Ind}(\text{add } M)$

Definition 3.1. Suppose \mathcal{C} is a full subcategory of $\Lambda\text{-mod}$. A Λ -module M is called Ext-projective in \mathcal{C} if $\text{Ext}_\Lambda^1(M, \mathcal{C}) = 0$. Dually, it is called Ext-injective in \mathcal{C} if $\text{Ext}_\Lambda^1(\mathcal{C}, M) = 0$.

The following lemma is from [AK].

Lemma 3.2. (1) $(\Lambda e)^\perp = {}^\perp(D(e\Lambda)) = \Lambda/\Lambda e\Lambda\text{-mod}$
(2) $(\text{Gen}(\Lambda e), \Lambda/\Lambda e\Lambda\text{-mod})$ and $(\Lambda/\Lambda e\Lambda\text{-mod}, \text{Cogen}(D(e\Lambda)))$ are both torsion pairs on $\Lambda\text{-mod}$.

Proof: It is clear that $\Lambda/\Lambda e\Lambda\text{-mod} = \{M \in \Lambda\text{-mod} \mid eM = 0\}$.

We claim: $(\Lambda e)^\perp = \{M \in \Lambda\text{-mod} \mid eM = 0\} = {}^\perp(D(e\Lambda))$.

In fact, for any $M \in (\Lambda e)^\perp$, $\text{Hom}(\Lambda e, M) = eM = 0$; For any $M \in {}^\perp(D(e\Lambda))$, $\text{Hom}(M, D(e\Lambda)) = \text{Hom}(M, \text{Hom}(e\Lambda, J)) = \text{Hom}(e\Lambda \otimes M, J) = D(eM) = 0 \iff eM = 0$.

By (1), (2) is clear.

Lemma 3.3. Let $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ be a 2-torsion pair on $\Lambda\text{-mod}$:

- (1) If $X \in \mathcal{C}_2$ is Ext-projective in $\langle \mathcal{C}_2, \mathcal{C}_3 \rangle$, $P_X \twoheadrightarrow X$ is the projective cover of X . Then, there exists an exact sequence $0 \rightarrow K_X \rightarrow P_X \rightarrow X \rightarrow 0$ such that $K_X \in \mathcal{C}_1$. Especially, $P_X \in \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ and $P_X \notin \mathcal{C}_1$.
- (2) If $Y \in \mathcal{C}_2$ is Ext-injective in $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$, $Y \hookrightarrow I_Y$ is the injective envelope of Y . Then, there exists an exact sequence $0 \rightarrow Y \rightarrow I_Y \rightarrow C_Y \rightarrow 0$ such that $C_Y \in \mathcal{C}_3$. Especially, $I_Y \in \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$ and $I_Y \notin \mathcal{C}_3$.

Proof: We only proof (1); The proof of (2) is similar.

By $(\mathcal{C}_1, \langle \mathcal{C}_2, \mathcal{C}_3 \rangle)$, there is a exact sequence $0 \rightarrow K_X \rightarrow P_X \rightarrow L \rightarrow 0$ such that $K_X \in \mathcal{C}_1$ and $L \in \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$, obviously, there is an epimorphism $\eta : L \rightarrow X$ if we apply $\text{Hom}(-, X)$ to the exact sequence. Since $\text{Ker } \eta \in \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$ and X is Ext-projective in $\langle \mathcal{C}_2, \mathcal{C}_3 \rangle$, η is split. Thus $L = X \oplus \text{Ker } \eta$, by the minimality of projective cover, $L = X$.

Lemma 3.4. *Let $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ be a 2-torsion pair on $\Lambda\text{-mod}$, and $X \in \Lambda\text{-mod}$ has a filtration*

$$\begin{array}{ccccccc} 0 & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 \longrightarrow X_3 \xlongequal{\quad} X \\ & & & & \swarrow & \swarrow & \swarrow \\ & & & & S_1 & S_2 & S_3 \end{array}$$

- (1) *If X is projective and $S_3 = 0$, then S_2 is Ext-projective in $\langle \mathcal{C}_2, \mathcal{C}_3 \rangle$ or $S_2 = 0$*
- (2) *If X is injective and $S_1 = 0$, then S_2 is Ext-injective in $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ or $S_2 = 0$*

Proof: We only proof (1); The proof of (2) is similar.

Since $S_3 = 0$, $X \cong X_3 \cong X_3$. Then $0 \rightarrow X_1 \rightarrow X \rightarrow S_2 \rightarrow 0$ is an exact sequence such that $X_1 \in \mathcal{C}_1, S_2 \in \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$. By Proposition 1.11 in Chapter 6 of [ASS], S_2 is Ext-projective in $\langle \mathcal{C}_2, \mathcal{C}_3 \rangle$.

Proposition 3.5. *Let $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ be an n -torsion pair on $\Lambda\text{-mod}$. Then there exists bijections:*

- (1) $F : \text{Ind } \mathcal{P}(\Lambda) \rightarrow \{X \in \text{Ind } \mathcal{C}_i \mid X \text{ is Ext-projective in } \langle \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle\};$
- (2) $G : \text{Ind } \mathcal{I}(\Lambda) \rightarrow \{Y \in \text{Ind } \mathcal{C}_j \mid Y \text{ is Ext-injective in } \langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_j \rangle\}.$

Proof: We only proof (1); The proof of (2) is similar.

Step 1. For any indecomposable projective Λ -module P , there is a filtration

$$\begin{array}{ccccccccccc} 0 & \xlongequal{\quad} & X_0 & \longrightarrow & \cdots & \longrightarrow & X_{i-1} & \longrightarrow & X_i & \longrightarrow & X_{i+1} & \longrightarrow & \cdots & \longrightarrow & X_{n+1} & \xlongequal{\quad} & P \\ & & & & & & \swarrow & & \swarrow & & \swarrow & & & & \swarrow & & \\ & & & & & & S_{i-1} & & S_i & & S_{i+1} & & & & S_{n+1} & & \end{array}$$

such that $0 \rightarrow X_i \rightarrow X_{i+1} \rightarrow S_{i+1} \rightarrow 0$ is an exact sequence and $S_i \in \mathcal{C}_i$.

Assume that $S_i \in \{S_1, S_2, \dots, S_{n+1}\}$ is the last non-zero module, then $S_{i+1} = \dots = S_{n+1} = 0$ and $X_i = X_{i+1} = \dots = X_{n+1} = P$.

Now, we consider the following filtration

$$\begin{array}{ccccccc} 0 & \xlongequal{\quad} & X_0 & \longrightarrow & X'_{i-1} & \longrightarrow & X_i \longrightarrow X_{i+1} \xlongequal{\quad} P \\ & & & & \swarrow & \swarrow & \swarrow \\ & & & & S'_{i-1} & S_i & S_{i+1} \end{array}$$

By lemma 3.4, S_i is Ext-projective in $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$. We denote $F(P) = S_i$.
Step2. Suppose $X \in \text{Ind } \mathcal{C}_i$ such that X is Ext-projective in $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$. Then we denote the projective cover of X By P_X and denote $F^{-1}(X) = P_X$.

Step 3. It is clear that $F^{-1}F(P) = P$ for any indecomposable projective module P . On the other hand, since $(\langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle, \mathcal{C}_i, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle)$ is a 3-torsion pair on $\Lambda\text{-mod}$, by Lemma 3.3, $FF^{-1}(X) = X$ for any $X \in \text{Ind } \mathcal{C}_i$ which is Ext-projective in $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$.

Corollary 3.6. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\Lambda\text{-mod}$. Then*

- (1) *there is an idempotent e such that $\mathcal{T} \cap \mathcal{P}(\Lambda) = \text{add } \Lambda e$, and $\mathcal{T} \cap (\Lambda e)^\perp$ has no Ext-projective modules in $(\Lambda e)^\perp$;*
- (2) *there is an idempotent e such that $\mathcal{F} \cap \mathcal{I}(\Lambda) = \text{add } D(e\Lambda)$, and ${}^\perp D(e\Lambda) \cap \mathcal{F}$ has no Ext-injective modules in ${}^\perp D(e\Lambda)$.*

Proof: We only proof (1); The proof of (2) is similar.

The first statement is clear, only the second one needs a proof:

$(\text{Gen}(\Lambda e), (\Lambda e)^\perp)$ is a torsion pair since Λe is a projective module. So we have a 2-torsion pair series $\{(\text{Gen}(\Lambda e), (\Lambda e)^\perp), (\mathcal{T}, \mathcal{F})\}$, and we have a 2-torsion pair $(\text{Gen}(\Lambda e), (\Lambda e)^\perp \cap \mathcal{T}, \mathcal{F})$.

Suppose that $X \in \mathcal{T} \cap (\Lambda e)^\perp$ is Ext-projective in $(\Lambda e)^\perp$. Then obviously, $X \notin \text{Gen}(\Lambda e)$. Let $f : P_X \twoheadrightarrow X$ is the projective cover of X . Then by proposition 3.4, $P_X \in \mathcal{T}$, and $X \in \text{Gen}(\Lambda e)$, this is a contradiction!

Lemma 3.7. *Let $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ be a 2-torsion pair on $\Lambda\text{-mod}$:*

- (1) *If $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ is closed under kernel, $X \in \mathcal{C}_1$ is Ext-projective in $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$, and $f : P_X \twoheadrightarrow X$ is the projective cover of X , then $P_X = X$ or $P_X \notin {}^\perp \mathcal{C}_3$;*
- (2) *If $\langle \mathcal{C}_2, \mathcal{C}_3 \rangle$ is closed under cokernel, $X \in \mathcal{C}_3$ is Ext-injective in $\langle \mathcal{C}_2, \mathcal{C}_3 \rangle$, and $g : X \hookrightarrow I_X$ is the injective envelope of X , then $I_X = X$ or $I_X \notin \mathcal{C}_1^\perp$.*

Proof: We only proof (1); The proof of (2) is similar.

Suppose $P_X \in {}^\perp \mathcal{C}_3 = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$, then exact sequence

$$0 \longrightarrow \text{Ker } f \longrightarrow P_X \xrightarrow{f} X \longrightarrow 0$$

is split in $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ since $X \in \mathcal{C}_1$ is Ext-projective in $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ and $\text{Ker } f \in \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$.

Corollary 3.8. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\Lambda\text{-mod}$.*

- (1) *If there are idempotents e^0, e^1 such that $\text{add } \Lambda e^0 \cap \text{add } \Lambda e^1 = 0$, $\mathcal{F} \cap \mathcal{I}(\Lambda) = \text{add } D(e^0 \Lambda)$, $\mathcal{T} \cap \mathcal{P}(\Lambda/\Lambda e^0 \Lambda) = \text{add } (\Lambda/\Lambda e^0 \Lambda)e^1$. Then $\mathcal{T} \cap \mathcal{P}(\Lambda) = \phi$ if and only if for any $P \in \text{add } \Lambda e^1$, $P \notin \Lambda/\Lambda e^0 \Lambda\text{-mod}$;*
- (2) *If there are orthogonal idempotents $\varepsilon^0, \varepsilon^1$ such that $\mathcal{T} \cap \mathcal{P}(\Lambda) = \text{add } \Lambda e^0$, $\mathcal{F} \cap \mathcal{I}(\Lambda/\Lambda \varepsilon^0 \Lambda) = \text{add } D(\varepsilon^1(\Lambda/\Lambda \varepsilon^0 \Lambda))$. Then $\mathcal{F} \cap \mathcal{I}(\Lambda) = \phi$ if and only if for any $I \in \text{add } D(\varepsilon^1 \Lambda)$, $I \notin \Lambda/\Lambda \varepsilon^0 \Lambda\text{-mod}$.*

Proof: We only proof (1); The proof of (2) is similar.

" \Rightarrow " Since $(\Lambda/\Lambda e^0\Lambda\text{-mod}, \text{Cogen}(D(e^0\Lambda)))$ is a torsion pair by lemma 3.2, $(\mathcal{T}, \mathcal{F} \cap \Lambda/\Lambda e^0\Lambda\text{-mod}, \text{Cogen}(D(e^0\Lambda)))$ is a 2 – torsion pair on $\Lambda\text{-mod}$. Suppose $0 \neq P \in \text{add } \Lambda e^1$. Then $P/e^0P \in \text{add}(\Lambda/\Lambda e^0\Lambda)e^1$ and $P/e^0P \neq 0$. So by the above lemma, $P = P/e^0P \in \mathcal{T}$ or $P \notin \Lambda/\Lambda e^0\Lambda\text{-mod}$. Since $\mathcal{T} \cap \mathcal{P}(\Lambda) = \phi$, $P \notin \Lambda/\Lambda e^0\Lambda\text{-mod}$.

" \Leftarrow " Suppose $\mathcal{T} \cap \mathcal{P}(\Lambda) \neq \phi$. Then there exists $0 \neq P \in \mathcal{T} \cap \mathcal{P}(\Lambda)$. Then P is also projective in $\Lambda/\Lambda e^0\Lambda\text{-mod}$. So $P \in \text{add } \Lambda e^1$. This is a contradiction.

Now we start to show the structure of torsion pairs by decomposing them by projective modules and injective modules. First we give some notations .

We always assume that $\Delta = \{e_1, e_2, \dots, e_n\}$ is a fixed complete set of primitive orthogonal idempotents of Λ . Given $S = \{\Delta_0, \Delta_1, \Delta_2, \dots, \Delta_m \mid \Delta_i \subseteq \Delta\}$ such that $\Delta_1, \Delta_2, \dots, \Delta_m \neq \phi$ and $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$, we have the following notations : $e_S^i = \sum_{e \in \Delta_i} e$, $\varepsilon_S^i = \sum_{j=0}^i e_S^j$; $\Lambda_S^0 = \Lambda$, $\Lambda_S^1 = \frac{\Lambda_S^0}{\Lambda_S^0 e_S^0 \Lambda_S^0} = \frac{\Lambda}{\Lambda e_S^0 \Lambda}$, \dots , $\Lambda_S^{m+1} = \frac{\Lambda_S^m}{\Lambda_S^m e_S^m \Lambda_S^m} = \frac{\Lambda}{\Lambda e_S^m \Lambda}$; $P_i(\Lambda_S^i) = \oplus_{e \in \Delta_i} \Lambda_S^i e$, $I_i(\Lambda_S^i) = \oplus_{e \in \Delta_i} D(e \Lambda_S^i)$.

Definition 3.9. Suppose S is as the above. It is called a 2-type part partition if:

(1) $\forall 0 < 2i \leq m$ and $e \in \Delta_{2i}$, $e_S^{2i-1} \Lambda_S^{2i-1} e \neq 0$; (2) $\forall 1 < 2i+1 \leq m$, and $e \in \Delta_{2i+1}$, $e \Lambda_S^{2i} e_S^{2i} \neq 0$.

Dually, S is called a 2-type part partition if: (1) $\forall 0 < 2i \leq m$ and $e \in \Delta_{2i}$, $e \Lambda_S^{2i-1} e_S^{2i-1} \neq 0$; (2) $\forall 1 < 2i+1 \leq m$, and $e \in \Delta_{2i+1}$, $e_S^{2i} \Lambda_S^{2i} e \neq 0$.

Lemma 3.10. Let I be an ideal of Λ , and e, e' be two idempotents. Then $\text{Hom}_{\Lambda/I}((\Lambda/I) \cdot e, D(e' \cdot \Lambda/I)) = 0$ if and only if $e' \cdot \Lambda/I \cdot e = 0$.

Proof: Notice that $\text{Hom}_{\Lambda/I}((\Lambda/I) \cdot e, D(e' \cdot \Lambda/I)) = D(e' \cdot \Lambda/I \cdot e)$.

We give the following notations for describing our theorem easily.

$\mathfrak{M} = \{(\mathcal{T}, \mathcal{F}) \mid (\mathcal{T}, \mathcal{F}) \text{ is a torsion pair on } \Lambda\text{-mod}\};$

$\mathfrak{N} = \{(S = \{\Delta'_0, \Delta'_1, \Delta'_2, \dots, \Delta'_m\}, (\mathcal{T}', \mathcal{F}')) \mid S \text{ is a 1-type part partition, } (\mathcal{T}', \mathcal{F}') \in \mathbf{E}(\Lambda_S^{m+1})\}.$

$\mathfrak{N}' = \{(S = \{\Delta'_0, \Delta'_1, \Delta'_2, \dots, \Delta'_m\}, (\mathcal{T}', \mathcal{F}')) \mid S \text{ is a 2-type part partition, } (\mathcal{T}', \mathcal{F}') \in \mathbf{E}(\Lambda_S^{m+1})\}.$

Now we are in a position to give a demonstration of how to decompose a torsion pair into n – torsion pair by projective modules and injective modules.

Let $(\mathcal{T}, \mathcal{F})$ be an torsion pair on $\Lambda\text{-mod}$:

a. Let $\mathcal{T}^0 = \mathcal{T}, \mathcal{F}^0 = \mathcal{F}, \Lambda^0 = \Lambda$, there exists some $\Delta_0 \subseteq \Delta$ such that $\mathcal{T}^0 \cap \mathcal{P}(\Lambda^0) = \text{add } \bigoplus_{e \in \Delta_0} \Lambda^0 e = \text{add } P_0(\Lambda^0)$. Let $\mathcal{T}^1 = \mathcal{T}^0 \cap (P_0(\Lambda^0))^\perp, \mathcal{F}^1 = \mathcal{F}^0, \Lambda^1 = \Lambda / \Lambda e^0 \Lambda$ where $e^0 = \sum_{e \in \Delta_0} e$. Then $(\mathcal{T}^1, \mathcal{F}^1)$ is a torsion pair on $\Lambda^1\text{-mod}$ and $\mathcal{T}^1 \cap \mathcal{P}(\Lambda^1) = \{0\}$ by corollary 3.6. Hence we have a 2 – torsion pair $(\text{Gen}P_0(\Lambda^0), \mathcal{T}^1, \mathcal{F}^1)$ on $\Lambda\text{-mod}$;

b. There exists some $\Delta_1 \subseteq \Delta - \Delta_0$ such that $\mathcal{F}^1 \cap \mathcal{I}(\Lambda^1) = \text{add } \bigoplus_{e \in \Delta_1} D(e\Lambda^1) = \text{add } I_1(\Lambda^1)$. Let $\mathcal{T}^2 = \mathcal{T}^1, \mathcal{F}^2 = \mathcal{F}^1 \cap {}^\perp I_1(\Lambda^1), \Lambda^2 = \Lambda / \Lambda \varepsilon^1 \Lambda$ where $\varepsilon^1 = \sum_{e \in \Delta_0 \cup \Delta_1} e$. Then $(\mathcal{T}^2, \mathcal{F}^2)$ is a torsion pair on $\Lambda^2\text{-mod}$ and $\mathcal{F}^2 \cap \mathcal{I}(\Lambda^2) = \{0\}$ by corollary 3.6. Hence we have a 3 – torsion pair $(\text{Gen}P_0(\Lambda^0), \mathcal{T}^2, \mathcal{F}^2, \text{Cogen}I_1(\Lambda^1))$ on $\Lambda\text{-mod}$;

The above operation goes on alternatively, then it will eventually stop since $\#\Delta$ is finite.

Finally, we obtain:

- (1) $\{\Delta_0, \Delta_1, \Delta_2, \dots, \Delta_m \mid \Delta_i \subseteq \Delta\}$ such that $\Delta_1, \Delta_2, \dots, \Delta_m \neq \emptyset$ and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$;
- (2) $(\mathcal{T}^{m+1}, \mathcal{F}^{m+1})$ is a torsion pair on $\Lambda^{m+1}\text{-mod}$ and $(\mathcal{T}^{m+1}, \mathcal{F}^{m+1}) \in \mathbf{E}(\Lambda^{m+1})$;
- (3) $(\text{Gen}P_0(\Lambda^0), \text{Gen}P_2(\Lambda^2), \dots, \mathcal{T}^{m+1}, \mathcal{F}^{m+1}, \dots, \text{Cogen}I_3(\Lambda^3), \text{Cogen}I_1(\Lambda^1))$ is a $(m+2)$ – torsion pair on $\Lambda\text{-mod}$;
- (4) $\Lambda = \Lambda^0 \rightarrow \Lambda^1 \rightarrow \dots \rightarrow \Lambda^{m+1}$ is a series of quotient algebras.

Theorem 3.11. *There is a one to one correspondence between \mathfrak{M} and \mathfrak{N} :*

$$\mathfrak{M} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \mathfrak{N}$$

Proof: Step 1. Suppose $(\mathcal{T}, \mathcal{F}) \in \mathfrak{M}$. we use the above operation. Then we get $S = \{\Delta_0, \Delta_1, \Delta_2, \dots, \Delta_m \mid \Delta_i \subseteq \Delta\}$ and $(\mathcal{T}^{m+1}, \mathcal{F}^{m+1}) \in \mathbf{E}(\Lambda^{m+1})$, so we need to prove S is a 2-type part partition, but it follows from corollary 3.6 and lemma 3.8. Let $F((\mathcal{T}, \mathcal{F})) = (S, (\mathcal{T}^{m+1}, \mathcal{F}^{m+1}))$.

Step 2. Suppose $(S = \{\Delta_0, \Delta_1, \Delta_2, \dots, \Delta_m\}, (\mathcal{T}', \mathcal{F}')) \in \mathfrak{N}$. By induction on m . It is easy to see that $(\text{Gen}P_0(\Lambda_S^0), \text{Gen}P_2(\Lambda_S^2), \dots, \mathcal{T}', \mathcal{F}', \dots, \text{Cogen}I_3(\Lambda_S^3), \text{Cogen}I_1(\Lambda_S^1))$ is a $(m+2)$ – torsion pair on $\Lambda\text{-mod}$. Let $G((S, (\mathcal{T}', \mathcal{F}'))) = (\mathcal{T}, \mathcal{F}) = (\langle \text{Gen}P_0(\Lambda_S^0), \text{Gen}P_2(\Lambda_S^2), \dots, \mathcal{T}' \rangle, \langle \mathcal{F}', \dots, \text{Cogen}I_3(\Lambda_S^3), \text{Cogen}I_1(\Lambda_S^1) \rangle)$.

Claim: $\mathcal{T} \cap \mathcal{P}(\Lambda) = \text{add } P_0(\Lambda^0)$.

Otherwise, there exists some $e \in \Delta - \Delta_0$ such that $\Lambda e \in \mathcal{T}$. By proposition 3.5 and the above $(m+2)$ – torsion pair, there exists $0 \neq X \in \text{Gen}P_{2i}(\Lambda_S^{2i})$ (or \mathcal{T}') for some $i \neq 0$, such that X is Ext-projective in Λ_S^{2i} (or Λ_S^{m+1}), and the projective cover of X is Λe since $\mathcal{T}' \cap \mathcal{P}(\Lambda_S^{m+1}) = \emptyset$ and $X \in P_{2i}(\Lambda_S^{2i})$. However, since

S is a 2-type part partition, $e_S^{2i-1}\Lambda_S^{2i-1}e \neq 0$. So $\text{Hom}_{\Lambda_S^{2i-1}}(X, D(e_S^{2i-1}\Lambda_S^{2i-1})) = \text{Hom}_{\Lambda_S^{2i-1}}(\Lambda_S^{2i-1}e, D(e_S^{2i-1}\Lambda_S^{2i-1})) \neq 0$. Hence $X \notin {}^\perp\mathcal{F}$. So $\Lambda e \notin {}^\perp\mathcal{F}$. A contradiction!

Step by step, we know $F(\mathcal{T}, \mathcal{F}) = (S, (\mathcal{T}', \mathcal{F}'))$.

Step 3. Given $(\mathcal{T}, \mathcal{F}) \in \mathfrak{M}$, it is clear that $GF(\mathcal{T}, \mathcal{F}) = (\mathcal{T}, \mathcal{F})$.

Dually, if we start to decompose a torsion pair from the right hand (torsion-free class), Then we have the following theorem :

Theorem 3.12. *There is a one to one correspondence between \mathfrak{M} and \mathfrak{N}' :*

$$\mathfrak{M} \begin{matrix} \xrightarrow{F'} \\ \xleftarrow{G'} \end{matrix} \mathfrak{N}'$$

It's natural to ask that what is the relation between the above two kinds of decomposition. The following theorem indicates that the decomposition of a torsion pair from left hand and right hand are the same.

Theorem 3.13. *Suppose $(\mathcal{T}, \mathcal{F}) \in \mathfrak{M}$, $F((\mathcal{T}, \mathcal{F})) = (S' = \{\Delta'_0, \Delta'_1, \Delta'_2, \dots, \Delta'_u\}, (\mathcal{T}', \mathcal{F}'))$ and $F'((\mathcal{T}, \mathcal{F})) = (S'' = \{\Delta''_0, \Delta''_1, \Delta''_2, \dots, \Delta''_v\}, (\mathcal{T}'', \mathcal{F}''))$. Then $(\mathcal{T}', \mathcal{F}') = (\mathcal{T}'', \mathcal{F}'')$.*

Proof: It is clear that $\mathcal{T}' = \mathcal{T} \cap \Lambda_{S'}^{u+1}\text{-mod}$, $\mathcal{F}' = \mathcal{F} \cap \Lambda_{S'}^{u+1}\text{-mod}$. And $(\mathcal{T}'', \mathcal{F}'')$ has the similar property. So we only need to prove $\Delta'_0 \cup \Delta'_1 \cup \dots \cup \Delta'_u = \Delta''_0 \cup \Delta''_1 \cup \dots \cup \Delta''_v$.

For convenience, we give the following notations for any given $i \geq 0$:

$$L_{S'}^i = \langle \text{GenP}_{2j}(\Lambda_{S'}^{2j}) \mid 0 \leq 2j \leq \max\{u, i\} \rangle;$$

$$R_{S'}^i = \langle \text{CogenI}_{2j+1}(\Lambda_{S'}^{2j+1}) \mid 0 \leq 2j+1 \leq \max\{u, i\} \rangle;$$

$$L_{S''}^i = \langle \text{GenP}_{2j+1}(\Lambda_{S''}^{2j+1}) \mid 0 \leq 2j+1 \leq \max\{v, i\} \rangle;$$

$$R_{S''}^i = \langle \text{CogenI}_{2j}(\Lambda_{S''}^{2j}) \mid 0 \leq 2j \leq \max\{v, i\} \rangle.$$

We just prove $\Delta'_0 \cup \Delta'_1 \cup \dots \cup \Delta'_u \subseteq \Delta''_0 \cup \Delta''_1 \cup \dots \cup \Delta''_v$. For this, we just need to prove: $\forall i \geq 0, L_{S'}^{2i+1} \subseteq L_{S''}^{2i+1}; R_{S'}^{2i} \subseteq R_{S''}^{2i}$.

For $i = 0, R_{S'}^0 = \{0\} \subseteq \text{CogenI}_0(\Lambda_{S''}^0) = R_{S''}^0, L_{S'}^1 = \text{GenP}_0(\Lambda_{S'}^0) \subseteq \text{GenP}_1(\Lambda_{S''}^1) = L_{S''}^1$.

Now we assume the theorem holds for $i \leq k-1$. Then $\Lambda_{S''}^{2k}$ is a quotient algebra of $\Lambda_{S'}^{2k-1}$ since $\Delta'_0 \cup \Delta'_1 \cup \dots \cup \Delta'_{2k-2} \subseteq \Delta''_0 \cup \Delta''_1 \cup \dots \cup \Delta''_{2k-1}$. So $\Lambda_{S''}^{2k}\text{-mod}$ is a full subcategory of $\Lambda_{S'}^{2k-1}\text{-mod}$.

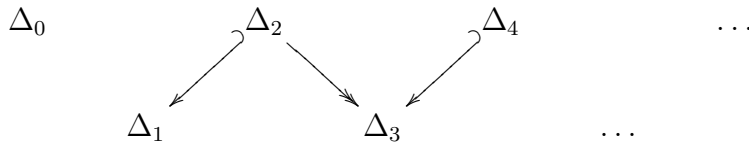
Suppose $0 \neq X \in \text{add I}_{2k-1}(\Lambda_{S'}^{2k-1})$. So X is Ext-injective in $\Lambda_{S''}^{2k}\text{-mod}$. By torsion pair $(\mathcal{F} \cap \Lambda_{S''}^{2k}\text{-mod}, R_{S''}^{2k-2})$ on \mathcal{F} , there exists an exact sequence $0 \rightarrow X_1 \rightarrow X \rightarrow X_2 \rightarrow 0$ such that $X_1 \in \mathcal{F} \cap \Lambda_{S''}^{2k}\text{-mod}$ and $X_2 \in R_{S''}^{2k-2}$. For every $Y \in$

$\Lambda_{S''}^{2k}$ -mod, applying $\text{Hom}_\Lambda(Y, -)$ to this exact sequence, we get an exact sequence: $\text{Hom}_\Lambda(Y, X_2) \rightarrow \text{Ext}_\Lambda^1(Y, X_1) \rightarrow \text{Ext}_\Lambda^1(Y, X)$. Since $\text{Ext}_\Lambda^1(Y, X) = 0$ and $Y \in {}^\perp(R_{S''}^{2k-2})$, $\text{Ext}_\Lambda^1(Y, X_1) = 0$. So X_1 is Ext-injective in $\Lambda_{S''}^{2k}$ -mod. Thus $X_1 \in \text{add } I_{2k}(\Lambda_{S''}^{2k})$. So $X \in R_{S''}^{2k}$. Therefore, $R_{S'}^{2k} \subseteq R_{S''}^{2k}$, and similarly, we have $L_{S'}^{2k+1} \subseteq L_{S''}^{2k+1}$.

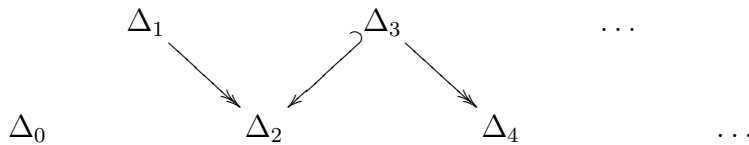
4 Examples

In this section, we will use the results developed in the previous two sections to characterize torsion pairs on some particular module categories. Those results will be related to [BBM], [BM], [HJR], [N], [HJ], [BK]. We always assume K is a field. If Q is a quiver and $\Delta \in Q_0$ where Q_0 is the set of vertices of Q , then we denote the full sub-quiver of Q containing Δ by $Q(\Delta)$. We give the following definition.

Definition 4.1. Let Q be a quiver, $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a tuple such that $\Delta_i \subseteq Q_0$, $\Delta_i \cap \Delta_j = \emptyset \forall i \neq j$, $\Delta_0 \neq \emptyset$. If $\forall i > 0$ and $v \in \Delta_{2i+1}$ there is a path from some vertex in Δ_{2i} to v in the sub-quiver $Q(Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_{2i-1})$, and $\forall i > 0$ and $v \in \Delta_{2i}$ there is a path from v to some vertex in Δ_{2i-1} in the sub-quiver $Q((Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_{2i-2}))$. Then we call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a 2-type part partition of Q . The following diagram shows the relation:



Dually, we call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a 2-type part partition of Q if $\forall i > 0$ and $v \in \Delta_{2i+1}$ there is a path from v to some vertex in Δ_{2i} in the sub-quiver $Q(Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_{2i-1})$, and $\forall i > 0$ and $v \in \Delta_{2i}$ there is a path from some vertex in Δ_{2i-1} to v in the sub-quiver $Q(Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_{2i-2})$. The following diagram shows the relation:



Epecially, if $\forall i > 0$, Δ_{2i-1} contains all sink points in $Q(Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_{2i-2})$, Δ_{2i} contains all source points in $Q(Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_{2i-1})$, then we call

$\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a strong 1-type part partition of Q . If $\forall i > 0$, Δ_{2i-1} contains all source points in $Q(Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_{2i-2})$, Δ_{2i} contains all sink points in $Q(Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_{2i-1})$, then we call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a strong 2-type part partition of Q .

If $\Delta_0 \cup \Delta_1 \cup \dots \cup \Delta_m = Q_0$ we call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a complete partition of Q .

We have the following lemma.

Lemma 4.2. *Let Q be a acyclic quiver and $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a strong 1-type part partition of Q . Then $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a 1-type part partition of Q .*

If $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a strong 2-type part partition of Q . Then $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a 2-type part partition of Q .

For a quiver Q , we denote $\mathbf{E}(KQ)$ by $\mathbf{E}(Q)$. Now we have the following theorem which is the path algebra's version of Theorem 3.11.

Theorem 4.3. *Let Q be a acyclic quiver. Then we have a bijection between the set $(\mathcal{T}, \mathcal{F})$ which is a torsion pair on $KQ\text{-mod}$ and the set of the pair $(\{\Delta_0, \Delta_1, \dots, \Delta_m\}; (\mathcal{T}', \mathcal{F}'))$, where $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a 1-type part partition of Q and $(\mathcal{T}', \mathcal{F}') \in \mathbf{E}(KQ(Q_0 - \Delta_0 - \Delta_1 - \dots - \Delta_m))$.*

The dual form of the theorem is similar, so we don't demonstrate here. Now let A_n be the following quiver: $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow n$. Applying the above theorem to the quiver A_n , we have the following theorem.

Theorem 4.4. *There exists a bijection between torsion pairs on $KA_n\text{-mod}$ and complete strong 1-type part partition sets of A_n .*

Proof: It is easy to see $\mathbf{E}(KA_m) = \phi$ for every m . And a complete partition of Q is a 2-type part partition if and only if it is strong 1-type part partition. The rest is clear by the above theorem.

If we observe the bijection above, then we obtain some simple corollaries.

Corollary 4.5. *Given a torsion pair $(\mathcal{T}, \mathcal{F})$ on $KA_n\text{-mod}$, then there exists a unique pair (T, F) such that T, F are basic partial tilting modules, $\# \text{Ind}(T \oplus F) = n$, and $\mathcal{T} = \text{Gen}(T), \mathcal{F} = \text{Cogen}(F)$.*

Corollary 4.6. *If $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a complete strong 1-type part partition of A_n , then the corresponding torsion pair is induced by tilting modules if and only if $v_1 \in \Delta_0$.*

If $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a complete strong 2-type part partition of A_n , then the corresponding torsion pair is induced by cotilting modules if and only if $v_n \in \Delta_0$.

Proposition 4.7. *The number of torsion pairs on KA_n is the $(n+1)$ -th Catalan number $C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$.*

Proof: Adding one vertex to A_n , then we have the quiver $A_{n+1} : 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow n \rightarrow n+1$. We have a torsion pair on $KA_{n+1}\text{-mod}$: $(KA_n\text{-mod}, \mathcal{P}(KA_{n+1}))$. So we have a bijection between torsion pairs on $KA_n\text{-mod}$ and torsion pairs induced by cotilting modules on $KA_{n+1}\text{-mod}$ by proposition 2.18. The number of torsion pairs induced by cotilting modules on $KA_{n+1}\text{-mod}$ is well known which is the $(n+1)$ -th Catalan number (Lemma A.1 in [BK]).

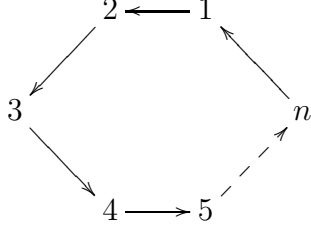
Definition 4.8. *Suppose Λ is an artin algebra, \mathcal{C} is a full subcategory of $\Lambda\text{-mod}$. If there exists a set of full subcategories $\{\mathcal{C}_i, i \in I\}$ of \mathcal{C} such that $\forall M \in \mathcal{C}$, there uniquely exists a set of modules $M_{i_1} \in \mathcal{C}_1, M_{i_2} \in \mathcal{C}_2, \dots, M_{i_n} \in \mathcal{C}_n$ where i_1, i_2, \dots, i_n are mutually different such that $M \cong M_{i_1} \oplus M_{i_2} \oplus \dots \oplus M_{i_n}$, then we call \mathcal{C} is the direct sum of $\{\mathcal{C}_i, i \in I\}$, and we denote $\mathcal{C} = \bigoplus_{i \in I} \mathcal{C}_i$.*

We have the following correspondence.

Lemma 4.9. *Suppose Λ is an artin algebra, \mathcal{C} is a full subcategory of $\Lambda\text{-mod}$, there exists a set of full subcategories $\{\mathcal{C}_i, i \in I\}$ of \mathcal{C} such that $\mathcal{C} = \bigoplus_{i \in I} \mathcal{C}_i$ and $\text{Hom}(X, Y) = 0$ for every $X \in \mathcal{C}_i, Y \in \mathcal{C}_j$ and $i \neq j$. Then there exists a bijection between torsion pairs on \mathcal{C} and the tuple $\{(\mathcal{T}_i, \mathcal{F}_i)\}_{i \in I}$ where $(\mathcal{T}_i, \mathcal{F}_i)$ is a torsion pair on \mathcal{C}_i .*

Proof: Given $(\mathcal{T}, \mathcal{F})$ a torsion pair on \mathcal{C} , then $(\mathcal{T} \cap \mathcal{C}_i, \mathcal{F} \cap \mathcal{C}_i)_{i \in I}$ is the corresponding tuple. Given the tuple $(\mathcal{T}_i, \mathcal{F}_i)_{i \in I}$ where $(\mathcal{T}_i, \mathcal{F}_i)$, then $(\bigoplus_{i \in I} \mathcal{T}_i, \bigoplus_{i \in I} \mathcal{F}_i)$ is the corresponding torsion pair.

Let \tilde{A}_n be the following quiver with vertices $(\tilde{A}_n)_0 = \{v_1, v_2, \dots, v_n\}$:



Let J be the ideal of $K\tilde{A}_n$ generated by all arrows. We call a finite-dimensional $K\tilde{A}_n$ module M is an ordinary module if there exists N such that $J^N M = 0$. In this condition M is a $K\tilde{A}_n/J^N$ module. So if M is indecomposable, then it is uniserial and determined by its socle and length. Let \mathcal{E}_n be the category of all ordinary modules. Then \mathcal{E}_n is closed under submodules, quotients and extensions. We denote the simple module corresponding to the vertex v_i by S_i . We will give all torsion pairs on \mathcal{E}_n . For this we give the following definition which is introduced in [BBM].

Definition 4.10. Suppose $\Delta \in (\tilde{A}_n)_0$. let $Ray(\Delta)$ be the category of all modules with socle in $\text{add } \bigoplus_{v_i \in \Delta} S_i$. let $Coray(\Delta)$ be the category of all modules with top in $\text{add } \bigoplus_{v_i \in \Delta} S_i$.

For a subcategory \mathcal{D} of \mathcal{E}_n . We denote $L_{\mathcal{D}}$ be the set of all vertices v_i such that there are infinite indecomposable modules in \mathcal{D} with S_i as the top, $R_{\mathcal{D}}$ be the set of all vertices v_j such that there are infinite indecomposable modules in \mathcal{D} with S_i as the socle .

By Definition 4.10 we have the following obvious lemma.

Lemma 4.11. Suppose $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$. Then $(Coray(\Delta), \tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}), (Q((\tilde{A}_n)_0 - \Delta)\text{-mod}, Ray(\Delta))$ are two torsion pairs on \mathcal{E}_n .

Now we give the following proposition.

Proposition 4.12. Suppose $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$. Then there is a bijection:

- (1) $\{(\mathcal{T}', \mathcal{F}') : \text{torsion pair on } \tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod which is induced by cotilting modules}\}$
 $\xrightleftharpoons[F']^F \{(\mathcal{T}, \mathcal{F}) : \text{torsion pair on } \mathcal{E}_n \text{ such that } L_{\mathcal{T}} = \Delta\}$. In this condition $F((\mathcal{T}', \mathcal{F}')) = (\langle Coray(\Delta), \mathcal{T}' \rangle, \mathcal{F}')$, $F'((\mathcal{T}, \mathcal{F})) = (\mathcal{T} \cap \tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}, \mathcal{F})$
- (2) $\{(\mathcal{T}', \mathcal{F}') : \text{torsion pair on } \tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod which is induced by tilting modules}\}$
 $\xrightleftharpoons[G']^G \{(\mathcal{T}, \mathcal{F}) : \text{torsion pair on } \mathcal{C} \text{ such that } R_{\mathcal{F}} = \Delta\}$. In this condition $G((\mathcal{T}', \mathcal{F}')) = (\mathcal{T}', \langle \mathcal{F}', Ray(\Delta) \rangle)$, $G'((\mathcal{T}, \mathcal{F})) = (\mathcal{T}, \mathcal{F} \cap \tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod})$.

Proof: We only proof (1) and (2) is similar.

(1). $\{(Coray(\Delta), \tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}), (\mathcal{E}_n, \{0\})\}$ is a 2-torsion pair seires on \mathcal{E}_n . Then by Proposition 2.18, we have a bijection between torsion pair $(\mathcal{T}, \mathcal{F})$ on \mathcal{E}_n such that $Coray(\Delta) \subseteq \mathcal{T}$ and torsion pairs on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}$. It is obvious in this condition $\Delta = L_{\mathcal{T}}$ if and only if in the corresponding torsion pair $(\mathcal{T}', \mathcal{F}')$ on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}$ \mathcal{F}' contains all projective modules which means it is induced by a cotilting module.

The following lemma is from Corollary 4.5 in [BBM].

Lemma 4.13. *Suppose $(\mathcal{T}, \mathcal{F})$ is a torsion pair on \mathcal{E}_n . Then $L_{\mathcal{T}}, R_{\mathcal{F}}$ are not both empty.*

Now we have the following theorem which gives all torsion pairs on \mathcal{E}_n .

Theorem 4.14. *The following are all mutually different torsion pairs on \mathcal{E}_n which are classified as two kinds.*

- (1) $(Coray(\Delta) \oplus \mathcal{T}', \mathcal{F}')$ for some $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$ and $(\mathcal{T}', \mathcal{F}')$ is a torsion pair on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}$ which is induced by cotilting modules.
- (2) $(\mathcal{T}', \mathcal{F}' \oplus Ray(\Delta))$ for some $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$ and $(\mathcal{T}', \mathcal{F}')$ is a torsion pair on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}$ which is induced by tilting modules.

Proof: Suppose $(\mathcal{T}, \mathcal{F})$ is a torsion pair on \mathcal{E}_n and $L_{\mathcal{T}} \neq \phi$. Then we know that $Coray(L_{\mathcal{T}}) \subseteq \mathcal{T}$ since \mathcal{T} is closed under quotients. And for the first kind it is obvious that $\langle Coray(\Delta), \mathcal{T}' \rangle = Coray(\Delta) \oplus \mathcal{T}'$. The other is similar.

Since $\phi \neq \Delta$, we know $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}$ is a direct sum of module categories of A_n -type algebras. so by Lemma 4.9 the torsion pair is easily obtained. By the above theorem and the characterization of torsion pairs induced by tilting or cotilting modules on A_n -type algebras, we have the following bijection.

Theorem 4.15. (1) *There is a bijection between the set of the torsion pairs $(\mathcal{T}, \mathcal{F})$ on \mathcal{E}_n such that $L_{\mathcal{T}} \neq \phi$ and the set of the complete sets of \tilde{A}_n $\{\Delta, \Delta_1, \dots, \Delta_m\}$ which is a strong 1-type part partition and Δ is not empty.*

(2) *There is a bijection between the set of the torsion pairs $(\mathcal{T}, \mathcal{F})$ on \mathcal{E}_n such that $R_{\mathcal{F}} \neq \phi$ and the set of the complete sest of \tilde{A}_n $\{\Delta, \Delta_1, \dots, \Delta_m\}$ which is a strong 2-type part partition and Δ is not empty.*

Proof: If $\{\Delta, \Delta_1, \dots, \Delta_m\}$ is a strong 1-type part partition, then $\{\Delta_1, \dots, \Delta_m\}$ is strong 1-type part partition in $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$. Then we get a torsion pair $(\mathcal{T}', \mathcal{F}')$ on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$ -mod which is induced by a cotilting module. Thus $(\text{Coray}(\Delta) \oplus \mathcal{T}', \mathcal{F}')$ is the corresponding torsion pair on \mathcal{E}_n

If $\{\Delta, \Delta_1, \dots, \Delta_m\}$ is a strong 2-type part partition, then we get a torsion pair $(\mathcal{T}', \mathcal{F}' \oplus \text{Ray}(\Delta))$ where $(\mathcal{T}', \mathcal{F}')$ is a torsion pair on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$ -mod which is induced by a tilting module.

The rest is clear.

5 Torsion pairs on hereditary algebras

In this section we always assume K is an algebraic closed field and Q is a acyclic quiver. We try to find a way to obtain all torsion pairs on KQ -mod. This aim is also the motivation of the article. If Q is not wild, we really get a way. If it is wild, the issue comes down to the torsion pairs on regular components of wild hereditary algebras. For this we denote the Auslander-Reiten translation by τ , its quasi-inverse by τ^- , the finite-dimensional projective KQ -module category by $\mathcal{P}(Q)$, the finite-dimensional injective KQ -module category by $\mathcal{I}(Q)$. The following two lemmas are well known.

Lemma 5.1. *Suppose $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence on kQ -mod. Then*

- (1) *If $\text{add } A \cap \mathcal{P}(Q) = \{0\}$, then $0 \rightarrow \tau A \rightarrow \tau B \rightarrow \tau C \rightarrow 0$ is an exact sequence.*
- (2) *If $\text{add } C \cap \mathcal{I}(Q) = \{0\}$, then $0 \rightarrow \tau^- A \rightarrow \tau^- B \rightarrow \tau^- C \rightarrow 0$ is an exact sequence.*

Lemma 5.2. *Suppose $X, Y \in kQ$ -mod.*

- (1) *If $\text{add } X \cap \mathcal{P}(Q) = \{0\}$, then $\text{Hom}(X, Y) \cong \text{Hom}(\tau X, \tau Y)$*
- (2) *If $\text{add } Y \cap \mathcal{I}(Q) = \{0\}$, then $\text{Hom}(X, Y) \cong \text{Hom}(\tau^- X, \tau^- Y)$*

We denote the set of torsion pairs on KQ -mod $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{I}(Q) \subseteq \mathcal{T}$ by $\mathbf{F}_1(Q)$ and the set of torsion pairs on KQ -mod $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{P}(Q) \subseteq \mathcal{F}$ by $\mathbf{F}_2(Q)$. And let $\mathbf{F}(Q) = \mathbf{F}_1(Q) \cup \mathbf{F}_2(Q)$. It is obvious that $\mathbf{E}(Q) = \mathbf{F}_1(Q) \cap \mathbf{F}_2(Q)$. As a consequence of the above two lemmas, we have the following proposition.

Proposition 5.3. *Suppose there is no projective-injective KQ -module. Then there is a one to one correspondence:*

$$\mathbf{F}_1(Q) \xrightleftharpoons[\sigma]{\sigma^-} \mathbf{F}_2(Q)$$

such that $\forall(\mathcal{T}', \mathcal{F}') \in \mathbf{F}_1(Q), \sigma^-(\mathcal{T}', \mathcal{F}') = (\tau^-\mathcal{T}', \tau^-\mathcal{F}' \oplus \mathcal{P}(Q)); \forall(\mathcal{T}'', \mathcal{F}'') \in \mathbf{F}_2(Q), \sigma(\mathcal{T}'', \mathcal{F}'') = (\mathcal{I}(Q) \oplus \tau\mathcal{T}'', \tau\mathcal{F}'')$.

Proof. We just prove that $\forall(\mathcal{T}', \mathcal{F}') \in \mathbf{F}_1, (\tau^-\mathcal{T}', \tau^-\mathcal{F}' \oplus \mathcal{P}(Q))$ is a torsion pair on $KQ\text{-mod}$.

By Lemma 5.2 (2), we know $\forall X \in \mathcal{T}', Y \in \mathcal{F}', \text{Hom}(\tau^-X, \tau^-Y) \cong \text{Hom}(X, Y) = \{0\}$. So the condition 1 in the Definition 2.1 is satisfied. By Lemma 5.1 (2), we know except projective modules, every indecomposable module has a suitable decomposition in $(\tau^-\mathcal{T}', \tau^-\mathcal{F}' \oplus \mathcal{P}(Q))$. But for projective modules, the suitable decomposition is obvious. So the condition 2 in the Definition 2.1 is satisfied.

Just like the Auslander-Reiten translation, σ^- and σ also gives a translation on $F(Q)$. For every $(\mathcal{T}, \mathcal{F}) \in \mathcal{F}(Q)$, if $\mathcal{I}(Q) \subseteq \mathcal{T}$, then let $\sigma^-(\mathcal{T}, \mathcal{F}) = (\tau^-\mathcal{T}, \tau^-\mathcal{F} \oplus \mathcal{P}(Q))$; if $\mathcal{P}(Q) \subseteq \mathcal{F}$, then let $\sigma(\mathcal{T}, \mathcal{F}) = (\tau\mathcal{T} \oplus \mathcal{I}(Q), \tau\mathcal{F})$. The above proposition tells us that this translation defines σ -orbits for elements in $\mathbf{F}(Q)$. We use $[\mathcal{T}, \mathcal{F}]$ to denote the σ -orbit of $(\mathcal{T}, \mathcal{F})$.

Definition 5.4. Suppose $(\mathcal{T}, \mathcal{F}) \in \mathbf{F}(Q)$. We call the elements in $[\mathcal{T}, \mathcal{F}] \cap (\mathbf{F}_2(Q) - \mathbf{F}_1(Q))$ source points of $[\mathcal{T}, \mathcal{F}]$, the elements in $[\mathcal{T}, \mathcal{F}] \cap (\mathbf{F}_1(Q) - \mathbf{F}_2(Q))$ sink points of $[\mathcal{T}, \mathcal{F}]$, the elements in $[\mathcal{T}, \mathcal{F}] \cap \mathbf{F}_1(Q) \cap \mathbf{F}_2(Q)$ middle points of $[\mathcal{T}, \mathcal{F}]$.

The following corollary is obvious.

Lemma 5.5. Suppose $(\mathcal{T}, \mathcal{F}) \in \mathbf{F}(Q)$. Then $[\mathcal{T}, \mathcal{F}]$ has at most one source point and at most one sink point. And $[\mathcal{T}, \mathcal{F}] \cap \mathbf{F}_1(Q) \cap \mathbf{F}_2(Q) = [\mathcal{T}, \mathcal{F}] \cap \mathbf{E}(Q)$.

We denote the preprojective component of $KQ\text{-mod}$ by $\mathcal{P}_\infty(Q)$, the preinjective component of $KQ\text{-mod}$ by $\mathcal{I}_\infty(Q)$, the regular component of $KQ\text{-mod}$ by $\mathcal{R}(Q)$.

Theorem 5.6. Suppose $(\mathcal{T}, \mathcal{F}) \in \mathbf{F}(Q)$. Then

- (1) $[\mathcal{T}, \mathcal{F}]$ has a source point but no sink point \iff for every $(\mathcal{T}', \mathcal{F}') \in [\mathcal{T}, \mathcal{F}]$, $\mathcal{I}_\infty(Q) \cap \mathcal{F}' \neq \phi$ and $\mathcal{P}_\infty(Q) \subseteq \mathcal{F}'$.
- (2) $[\mathcal{T}, \mathcal{F}]$ has a sink point but no source point \iff for every $(\mathcal{T}', \mathcal{F}') \in [\mathcal{T}, \mathcal{F}]$, $\mathcal{P}_\infty(Q) \cap \mathcal{T} \neq \phi$ and $\mathcal{I}_\infty(Q) \subseteq \mathcal{T}'$.
- (3) $[\mathcal{T}, \mathcal{F}]$ has a sink point and a source point \iff for every $(\mathcal{T}', \mathcal{F}') \in [\mathcal{T}, \mathcal{F}]$, $\mathcal{I}_\infty(Q) \cap \mathcal{F} \neq \phi$ and $\mathcal{P}_\infty(Q) \cap \mathcal{T} \neq \phi$.

(4) $[\mathcal{T}, \mathcal{F}]$ has no sink point and no source point \iff for every $(\mathcal{T}', \mathcal{F}') \in [\mathcal{T}, \mathcal{F}]$, $\mathcal{I}_\infty(Q) \subseteq \mathcal{T}'$, and $\mathcal{P}_\infty(Q) \subseteq \mathcal{F}'$.

We denote the set of torsion pairs $(\mathcal{T}, \mathcal{F})$ on $KQ\text{-mod}$ such that $\mathcal{I}_\infty(Q) \subseteq \mathcal{T}$, and $\mathcal{P}_\infty(Q) \subseteq \mathcal{F}$ by $\mathbf{H}(Q)$. So it is obvious that $\mathbf{H}(Q) \subseteq \mathbf{E}(Q)$. We denote the set of torsion pairs on $\mathcal{R}(Q)$ by $\mathbf{R}(Q)$. We have the following obvious lemma.

Lemma 5.7. *There is a one to one correspondence:*

$$\mathbf{H}(Q) \begin{matrix} \xrightarrow{F} \\ \xleftarrow{F^-} \end{matrix} \mathbf{R}(Q)$$

such that $\forall (\mathcal{T}, \mathcal{F}) \in \mathbf{H}(Q)$, $F((\mathcal{T}, \mathcal{F})) = (\mathcal{T} \cap \mathcal{R}(Q), \mathcal{F} \cap \mathcal{R}(Q))$; $\forall (\mathcal{T}', \mathcal{F}') \in \mathbf{R}(Q)$, $F^-((\mathcal{T}', \mathcal{F}')) = (\mathcal{T}' \oplus \mathcal{I}_\infty(Q), \mathcal{F}' \oplus \mathcal{P}_\infty(Q))$.

Remark 5.8. *Suppose $(\mathcal{T}, \mathcal{F}) \in \mathbf{F}(Q)$ and $[\mathcal{T}, \mathcal{F}]$ has at least one sink point or one source point. We define the following operation Φ :*

Case 1. If $[\mathcal{T}, \mathcal{F}]$ has a sink point, then we denote the sink point by $\Phi((\mathcal{T}, \mathcal{F}))$.

Case 2. If $[\mathcal{T}, \mathcal{F}]$ has a source point but no sink point, then we denote the source point by $\Phi((\mathcal{T}, \mathcal{F}))$.

For any torsion pair on $KQ\text{-mod}$ we apply the operation in Theorem 3.11 and the operation Φ to it alternatively. At last we get a new torsion pair on $KQ'\text{-mod}$ for some subquiver Q' of Q such that the new torsion pair belongs to $\mathbf{H}(Q')$. This process is invertible by Theorem 3.11 and Proposition 5.3. So by the above lemma if we know all torsion pairs on regular components for all subquivers, then we can construct all torsion pairs of $KQ\text{-mod}$.

From now on we suppose Q is a acyclic quiver with a Euclid ground graph. We start to find all the torsion pairs on $\mathbf{R}(Q)$. The following definition and two lemmas are from [WB].

Definition 5.9. *Suppose $X \in KQ\text{-mod}$. Then Q is regular uniserial if there are regular submodules $0 = X_0 \subset X_1 \subset \cdots \subset X_r = X$ and these are the only regular submodules of X .*

Lemma 5.10. *If $\theta : X \rightarrow Y$ with X, Y regular $KQ\text{-modules}$, then $\text{Im}(\theta)$, $\text{Ker}(\theta)$ and $\text{Coker}(\theta)$ are regular.*

Lemma 5.11. *Every indecomposable regular $KQ\text{-module}$ is regular universal.*

As an consequence we have

Corollary 5.12. *If KQ is an Euclid-type algebra, X is a regular module, then the quotient modules of X forms a chain: $X = X^r \twoheadrightarrow \cdots \twoheadrightarrow X^1 \twoheadrightarrow X^0$.*

Corollary 5.13. *Let KQ be an Euclid-type algebra, $f : X \rightarrow Y$ is an injective morphism such that X is a maximal regular submodule of the indecomposable regular module of Y . Then f is an irreducible morphism.*

Proof. X is indecomposable by Lemma 5.11. Suppose $\exists g : X \rightarrow Z, h : Z \rightarrow Y$ such that $f = hg$. Then by Lemma 5.11, there is an indecomposable direct summand Z' such that $\exists g' : X \rightarrow Z', h : Z' \rightarrow Y$ such that $h'g'$ is an injective morphism. Z' is a regular module. So by Lemma 5.11, h' is an injective morphism. Since X is a maximal regular submodule, h' is an isomorphism or g' is an isomorphism.

Now Let $\mathcal{R}(Q) = \bigoplus_{i \in I} \mathcal{R}_i(Q)$ where $\{\mathcal{R}_i(Q), i \in I\}$ is the set of minimal additive categories containing a connected component in AR-quiver of KQ . We denote the set of torsion pairs on $\mathcal{R}_i(Q)$ by $\mathbf{R}_i(Q)$. By Lemma 4.9, we have the following lemma.

Corollary 5.14. *There exists a bijection between $\mathbf{R}(Q)$ and the set of tuples $\{(\mathcal{T}_i, \mathcal{F}_i)\}_{i \in I}$ with $(\mathcal{T}_i, \mathcal{F}_i) \in \mathbf{R}_i(Q)$.*

Proof: Let $X \in \mathcal{R}_i(Q)$. Then all regular submodules and all regular quotient modules of X are in $\mathcal{R}_i(Q)$ by the above corollary. So we know if $i \neq j$, then $\text{Hom}(X, Y) = 0, \forall X \in \mathcal{R}_i(Q)$ and $Y \in \mathcal{R}_j(Q)$. The rest is clear by Lemma 4.9.

Now we start to demonstrate $\mathbf{R}_i(Q)$. Suppose $\mathbf{R}_i(Q)$ has n regular simple modules[WB]: S_1, S_2, \dots, S_{n-1} where $S_{i+1} = \tau S_i$. Let \tilde{A}_n be the quiver in Section 4 and S'_1, S'_2, \dots, S'_n are the correspondent simple modules to the vertices. Then we construct a map: $\overline{F}(S'_i) = S_i$. Then \overline{F} induces a one to one correspondence: $\mathcal{E}_n \rightarrow \mathcal{R}_i(Q)$ such that if $X \in \mathcal{E}_n$ and is indecomposable with the length m and top S'_i , then $F(X)$ is the indecomposable regular module with the regular length m and top S_i . we have the following lemma.

Lemma 5.15. (1) $\forall X, Y \in \mathcal{E}_n, \text{Hom}(X, Y) = 0 \iff \text{Hom}(F(X), F(Y)) = 0$.
(2) Suppose $Y \in \mathcal{E}_n$ and X is a submodule of Y . Then $F(Y/X) = F(Y)/F(X)$.

Proof: Clear by Lemma 5.11 and Corollary 5.12.

Theorem 5.16. *F induces a one to one correspondent between the set of torsion pairs on \mathcal{E}_n and $\mathbf{R}_i(Q)$.*

Proof: Clear by the above lemma.

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